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Recursion relations for tree amplitudes in super gauge theories

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Abstract: Using newly proposed recursion relations in [25, 26], compact formulas are obtained for tree-level amplitudes of $g_1^- g_2^- g_3^- g_4^+ g_5^+ \cdots g_n^+$. We then make an extension of these recursion relations to include fermions of multi-flavors, from which MHV and $\overline{\text{MHV}}$ amplitudes are reproduced. We also calculate non-MHV amplitudes of processes with two fermions and four gluons. Results thus obtained are equivalent to those obtained by extended CSW prescriptions, and those by conventional field theory calculations.

Keywords: Gauge Symmetry, Extended Supersymmetry, QCD.

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Contents

1. Introduction

In perturbative gauge theories, extremely simple results usually emerge after tremendously complicated calculations by using conventional field theory methods. For example, the Parke-Taylor formula for maximal helicity violating (MHV) amplitudes can be written within one line, which summarizes numerous ordinary Feynman diagrams [1].

Recently, a deep relation was pointed out between $N = 4$ super gauge theory and one type B topological string theory [2], by re-expressing super gauge theory scattering amplitudes in the language of twistor theories [3]. Taking advantage of insights thus gained and by a careful analysis of known helicity amplitudes, Cachazo, Svrcek, and Witten (CSW) [4] proposed a novel prescription to calculate tree level amplitudes, which uses MHV amplitudes as vertices to construct all other amplitudes, thus initiated the paradigm of MHV Feynman diagrams. The efficiency of the method is phenomenal and its validity has been checked by various tree level calculations [5]–[13]. In [14], the method was extended to calculate one-loop MHV amplitudes and correct results were reproduced in the large N_c limit. It turns out that the paradigm is independent of the large N_c approximation, at least to one-loop [15, 16]. The twistor-space structure of one-loop amplitudes are further studied in [17]–[20]. On the other hand, tree-level amplitudes were also obtained from connected curves in twistor string theories [21, 22, 23].

However, the number of MHV diagrams grows rapidly when one includes more external particles, though real amplitudes may well be much simpler than those suggested by CSW rules. For example, an extremely simple result was obtained for an amplitude of eight gluons in [24], by dissolving $N = 4$ loop amplitudes into tree ones. Related, Britto, Cachazo and Feng (BCF) proposed a new set of recursion relations to calculate tree amplitudes [25], based upon analysis of one-loop amplitudes and infrared relations. The new method has reproduced known results up to seven external gluons, and can easily be applied to processes with any reasonable number of external gluons. Soon after, the BCF prescription was proved alternatively by using basic facts of tree diagrams with some help from MHV Feynman diagrams [26] (BCFW). In [25], the recursion relations used two adjacent gluons of opposite helicity as reference gluons. However, it has been shown in [26] that reference gluons do not have to be adjacent and they can also be of the same helicity. Related works can be found in [27, 28].

As a concrete application of the BCF prescription, we will in this paper calculate tree-level amplitudes of $g_1^- g_2^- g_3^- g_4^+ g_5^+ \cdots g_n^+$. As expected, extremely compact formulas are obtained, which will be presented in section 2 after a brief review of the BCF rules. These recursion relations are then extended to include fermions of multi-flavors, in the context of super gauge theories, from which MHV and $\overline{\mathrm{MHV}}$ amplitudes are reproduced correctly (section 3). In section 4, we will calculate non-MHV amplitudes of processes with two fermions and four gluons, by using of both our extensions and extended CSW prescriptions [12]. Results obtained by these two methods are equivalent to each other, and equivalent to those given in [29] by conventional field theory calculations. These results provide strong evidences to support both the CSW and the BCF/BCFW prescriptions, as well as the present extension.

2. Review of the BCF/BCFW approach and a compact formula

In four dimensional space-time, a four-vector p_μ can always be expressed as a bispinor $p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}}$, and a null vector can be factorized $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ in terms of spinors λ_a, $\tilde{\lambda}_{\dot{a}}$ of positive and negative chirality. Spinor products are defined to be $\langle \lambda_1, \lambda_2 \rangle = \epsilon_{ab} \lambda_1^a \lambda_2^b$ and $\langle \tilde{\lambda}_1, \tilde{\lambda}_2 \rangle = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_1^{\dot{a}} \tilde{\lambda}_2^{\dot{b}}$, which are usually abbreviated as $\langle 1, 2 \rangle$ and $[1, 2]$. For two null vectors $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ and $q_{a\dot{a}} = \mu_a \tilde{\mu}_{\dot{a}}$, one has $2p \cdot q = \langle \lambda, \mu \rangle [\tilde{\lambda}, \tilde{\mu}]$.

Now we give a brief review of the BCF/BCFW approach. Take an n-gluon tree-level amplitude of any helicity configuration. As amplitudes of gluons with one helicity vanish, we can always arrange gluons such that the $(n-1)$-th gluon has negative helicity and the n-th gluon has positive helicity. These two lines will be taken as reference lines. Labeling external particles by i, the following recursion relation was claimed in [25]:

$$A_n\left(1, 2, \ldots, (n-1)^-, n^+\right) = \sum_{i=1}^{n-3} \sum_{h=+,-} \Big(A_{i+2}(\hat{n}, 1, 2, \ldots, i, -\hat{P}^h_{n,i}) \times \\ \times \frac{1}{P^2_{n,i}} A_{n-i}(+\hat{P}^{-h}_{n,i}, i+1, \ldots, n-2, \widehat{n-1}) \Big) \tag{2.1}$$

where

$$\begin{aligned} P_{n,i} &= p_n + p_1 + \ldots + p_i\,, \\ \hat{P}_{n,i} &= P_{n,i} + \frac{P^2_{n,i}}{\langle n-1 | P_{n,i} | n]} \lambda_{n-1} \tilde{\lambda}_n\,, \\ \hat{p}_{n-1} &= p_{n-1} - \frac{P^2_{n,i}}{\langle n-1 | P_{n,i} | n]} \lambda_{n-1} \tilde{\lambda}_n\,, \\ \hat{p}_n &= p_n + \frac{P^2_{n,i}}{\langle n-1 | P_{n,i} | n]} \lambda_{n-1} \tilde{\lambda}_n\,. \end{aligned} \tag{2.2}$$

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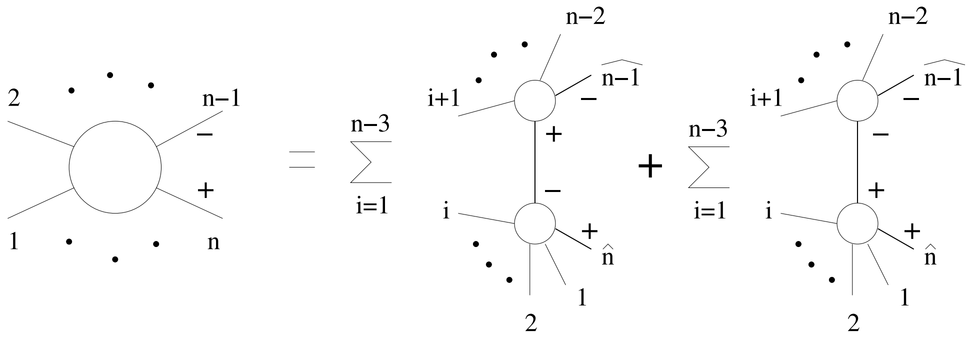

Figure 1: Pictorial representation of the recursion relation (2.1). Note that the difference between the terms in the two sums is just the helicity assignment of the internal line [25].

The formula has a natural meaning in $(--++)$ signature and can be pictorially represented, as shown in figure 1. But the recursion relations do not depend the signature and we will take it to be $(+---)$. Notice that $\hat{P}^2_{n,i} = \hat{p}^2_n = \hat{p}^2_{n-1} = 0$, so each tree-level amplitude in equation (2.1) has all external gluons on-shell. Still, energy-momentum conservation is preserved. In [26], it has been shown that reference gluons do not have to be adjacent and they can be of the same helicity. As we shall see late, one could get equivalent but more compact results if reference particles are of the same helicity, or the other way around.

To streamline notations, it is expedient to define

$$\begin{aligned} K_i^{[r]} &\equiv p_i + p_{i+1} + \cdots + p_{i+r-1} \\ t_i^{[r]} &\equiv \left(K_i^{[r]}\right)^2 \\ \left\langle i|\sum_r p_r|j\right] &\equiv \sum_r \langle i\ r\rangle [r\ j] \\ \left\langle i|(\sum_r p_r)(\sum_s p_s)|j\right\rangle &\equiv \sum_r \sum_s \langle i\ r\rangle [r\ s]\langle s\ j\rangle \end{aligned} \tag{2.3}$$

And the following relations will be useful to make simplifications:

$$\langle A|\hat{P}_{n,i}\rangle = -\frac{1}{\omega}\langle A|P_{n,i}|n] , \qquad [\,\hat{P}_{n,i}|B] = -\frac{1}{\bar{\omega}}\langle n-1|P_{n,i}|B] . \tag{2.4}$$

Factors ω and $\bar{\omega}$ always show up in final results in the combination of $\omega\bar{\omega}$, which is equal to $\langle n-1|P_{n,i}|n\rangle$. This completes the BCF/BCFW prescription.

Now, we use these rules to calculate tree-level amplitudes of $g_1^- g_2^- g_3^- g_4^+ g_5^+ \cdots g_n^+$. After some experimentation, the following general result for arbitrary n can be guessed

$$\begin{aligned} A(1^-2^-3^-4^+\cdots n^+) = {} & \frac{\langle 1|K_2^{[2]}|4]^3}{t_2^{[3]}[2\ 3][3\ 4]\langle 5|K_3^{[2]}|2]\prod_{i=5}^n \langle i\ i+1\rangle} - \\ & -\sum_{i=2}^{n-5} \frac{\langle 3|K_4^{[i]}K_{4+i}^{[n-3-i]}|1\rangle^3}{t_3^{[i+1]}t_2^{[i+2]}\langle i+3|K_3^{[i]}|2]\langle i+4|K_3^{[i+1]}|2]} \frac{\langle i+3\ i+4\rangle}{\prod_{j=3}^n \langle j\ j+1\rangle} + \\ & + \frac{\langle 3|K_1^{[2]}|n]^3}{t_n^{[3]}[n\ 1][1\ 2]\langle n-1|K_n^{[2]}|2]\prod_{j=3}^{n-2}\langle j\ j+1\rangle} \end{aligned} \tag{2.5}$$

Its validity can be easily proved by induction with the help of BCF rules. On the other hand, one gets the above result directly if g_1^-, g_2^- are chosen as reference lines. These results are much compact than those given in [8, 30].

3. Extensions to include fermions, MHV and $\overline{\text{MHV}}$ amplitudes

The BCF rules can be naturally extended to include fermions. Specifically, one chooses any two external lines, either gluons or fermions, as reference lines; then shifts relevant momenta exactly in the same manner as those in equation (2.2) and finally, combines sub-amplitudes of less numbers of external lines together in the same manner as that in equation (2.1). Of course, there could now be diagrams in which sub-diagrams are linked by internal fermionic lines, but their propagators are the same as those of gluons. For processes with different flavors of fermions, one should properly exclude diagrams in which fermions would be forced to change flavors in sub-diagrams. For example, in a process of four fermions of two types, $(\Lambda_1^-, \Lambda_2^+, \eta_3^-, \eta_4^+)$, one should exclude u-channel diagrams. These extensions can be justified by similar reasonings as those in either [25] or [26], and by careful analysis of ordinary Feynman diagrams.

But there are some qualifications. Specifically, one can take neither two adjacent fermion lines of the same type and opposite helicities, nor two adjacent fermion lines of different types and the same helicity, nor one fermion and an adjacent gluon of the same helicity, as reference lines. Otherwise one runs into inconsistencies. Actually, the rational function $A(z)$ defined in equation (2.3) of [26] would not vanish as $z \to \infty$ with such a choice, which spoils the recursion relations. We shall come back to this later.

Now we prove by induction that these new rules generate correct MHV and $\overline{\text{MHV}}$ amplitudes with two or four fermions. As reference lines can be chosen as wish, there are numerous but equivalent proofs. Details of each proof also depend on relative positions between fermions and gluons. One has to prove each case by exhausting all possibilities. However, all proofs are similar and a full list would be tedious and dull. We will instead give below three representative cases for illustration. Other cases can be worked out accordingly.

The first example has the following structure, $(g_1^-, g_2^+, \ldots, \Lambda_i^-, \ldots, \Lambda_j^+, \cdots, g_n^+)$ $(i > 3)$, which has two fermions. Assume generalized Parke-Taylor formulas for MHV amplitudes with fermions of $i < n$ external particles and pick reference lines to be g_1^-, g_2^+, we have

$$\frac{\langle \widehat{1}\ i\rangle^3 \langle \widehat{1}\ j\rangle}{\langle \widehat{1}\ \widehat{p}\rangle \langle \widehat{p}\ 4\rangle \langle n\ \widehat{1}\rangle \prod_{m=4}^{n-1} \langle m\ m+1\rangle} \times \frac{1}{t_2^{[2]}} \times \frac{[\widehat{2}\ 3]^3}{[3\ \widehat{p}][\widehat{p}\ \widehat{2}]} = \frac{\langle 1\ i\rangle^3 \langle 1\ j\rangle}{\prod_{m=1}^{n} \langle m\ m+1\rangle} \tag{3.1}$$

which is just the generalized Parke-Taylor formula for MHV amplitudes with fermions of n external particles. The second example has four fermions of the same flavor and the following structure, $(\Lambda_1^-, \Lambda_i^+, \Lambda_j^-, \Lambda_k^+)$ $(i > 2,\ j > i+1)$ and the reference lines will be taken to be Λ_1^-, Λ_i^+. It turns out that there are two terms. The first one is

$$A_1 = \frac{[i+1\ \widehat{i}]^3 [i+1\ \widehat{p}]}{[i+1\ \widehat{p}][\widehat{p}\ \widehat{i}][\widehat{i}\ i+1]} \times \frac{1}{t_i^{[2]}} \times$$

$$\times \frac{\langle \hat{1}\ j\rangle^3 \langle \hat{p}\ k\rangle}{\langle \hat{p}, i+2\rangle \langle n\ \hat{1}\rangle \langle \hat{1}\ 2\rangle \langle i-1\ \hat{p}\rangle \prod_{m=i+2}^{n-1} \langle m\ m+1\rangle \prod_{r=2}^{i-2} \langle r\ r+1\rangle} =$$

$$= \frac{\langle k\ i+1\rangle \langle i-1\ i\rangle}{\langle k\ i\rangle \langle i-1\ i+1\rangle} \frac{\langle 1\ j\rangle^3 \langle i\ k\rangle}{\prod_{m=1}^{n} \langle m\ m+1\rangle} \tag{3.2}$$

and the second one is

$$A_2 = \frac{[i-1\ \hat{i}]^3 [i-1\ \hat{p}]}{[i-1\ \hat{i}][\hat{i}\ \hat{p}][\hat{p}\ i-1]} \times \frac{1}{t_{i-1}^{[2]}} \times$$

$$\times \frac{\langle \hat{1}\ j\rangle^3}{\langle \hat{1}\ 2\rangle \langle i-2\ \hat{p}\rangle \langle \hat{p}\ i+1\rangle \prod_{m=2}^{i-3} \langle m\ m+1\rangle \prod_{r=i+1}^{n} \langle r\ r+1\rangle} =$$

$$= \frac{\langle k\ i-1\rangle \langle i\ i+1\rangle}{\langle k\ i\rangle \langle i-1\ i+1\rangle} \frac{\langle 1\ j\rangle^3 \langle i\ k\rangle}{\prod_{m=1}^{n} \langle m\ m+1\rangle} \tag{3.3}$$

Using Schouten's identity

$$\langle i\ i+1\rangle \langle k\ i-1\rangle + \langle i-1\ i\rangle \langle k\ i+1\rangle = \langle i\ k\rangle \langle i+1\ i-1\rangle \tag{3.4}$$

to combine them together, we have

$$A(\Lambda_1^- \Lambda_i^+ \Lambda_j^- \Lambda_k^+) = \frac{\langle 1\ j\rangle^3 \langle i\ k\rangle}{\prod_{m=1}^{n} \langle m\ m+1\rangle} \tag{3.5}$$

which is the generalized Parke-Taylor formula for MHV amplitudes with four fermions of the same type. The third example has four fermions of two flavors and the following structure, $(\Lambda_1^-, \Lambda_i^+, \eta_j^-, \eta_k^+)$, if the reference lines are chosen to be Λ_1^- and g_2^+, one then has inductively

$$A(\Lambda_1^- \Lambda_i^+ \eta_j^- \eta_k^+) = \frac{[\hat{2}\ 3]^3}{[3\ \hat{p}][\hat{p}\ \hat{2}]} \times \frac{1}{t_2^{[2]}} \times \frac{\langle \hat{1}\ j\rangle^2 \langle i\ j\rangle \langle \hat{1}\ k\rangle}{\langle \hat{1}\ \hat{p}\rangle \langle \hat{p}\ 4\rangle \langle n\ \hat{1}\rangle \prod_{m=4}^{n-1} \langle m\ m+1\rangle}$$

$$= \frac{\langle 1\ j\rangle^2 \langle i\ j\rangle \langle 1\ k\rangle}{\prod_{m=1}^{n} \langle m\ m+1\rangle} \tag{3.6}$$

which is the generalized Parke-Taylor formula for MHV amplitudes with four fermions of two flavors. Thus complete inductions for these three cases.

As indicated above, other cases can be worked out similarly. Almost identical reasonings lead to the same conclusion for $\overline{\text{MHV}}$ amplitudes. In a sense, these results can be anticipated. As pointed in [25], the BCF rules yield the correct MHV amplitudes of pure gluons. On the other hand, $\overline{\text{MHV}}$ amplitudes of pure gluons can be also obtained inductively, by following the same steps. From them, MHV and $\overline{\text{MHV}}$ amplitudes with two or four fermions can be obtained via Ward identities. But proofs outlined above are welcome checks for self-consistency of our extensions.

4. Scattering amplitudes of $\Lambda_1^+ \Lambda_2^- g_3 g_4 g_5 g_6$

Now we apply the rules in the previous section to calculate non-MHV amplitudes for processes with two fermions and four gluons. Paralleling results in [29], we calculate the following six amplitudes.

Choosing p_5 and p_6 as reference momenta, we have

$$A(\Lambda_1^+\Lambda_2^- g_3^- g_4^+ g_5^- g_6^+) = -\frac{\langle 3\ 2\rangle^2\langle 3\ 1\rangle[4\ 6]^4}{t_1^{[3]}\langle 1\ 2\rangle[4\ 5][5\ 6]\langle 3|4+5|6]\langle 1|5+6|4]} - \\ -\frac{\langle 1\ 5\rangle[2\ 4]\langle 5|1+6|4]^3}{t_2^{[3]}\langle 1\ 6\rangle\langle 5\ 6\rangle[2\ 3][3\ 4]\langle 1|5+6|4]\langle 5|1+6|2]} - \\ -\frac{\langle 3\ 5\rangle^4[6\ 1]^2[6\ 2]}{t_3^{[3]}\langle 3\ 4\rangle\langle 4\ 5\rangle[1\ 2]\langle 3|1+2|6]\langle 5|1+6|2]} \tag{4.1}$$

Choosing p_4 and p_5 as reference momenta, we have

$$A(\Lambda_1^+\Lambda_2^- g_3^- g_4^- g_5^+ g_6^+) = -\frac{\langle 4|5+6|1]^2\langle 4|5+6|2]}{t_4^{[3]}[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle[3|4+5|6\rangle} + \\ +\frac{\langle 2|3+4|5]^2\langle 1|3+4|5]}{t_3^{[3]}\langle 1\ 2\rangle\langle 6\ 1\rangle[3\ 4][4\ 5][3|4+5|6\rangle} \tag{4.2}$$

Choosing p_5 and p_6 as reference momenta, we have

$$A(\Lambda_1^+\Lambda_2^- g_3^+ g_4^- g_5^- g_6^+) = -\frac{\langle 1\ 5\rangle\langle 5|1+6|3]^3}{t_2^{[3]}\langle 1\ 6\rangle[3\ 4]\langle 5\ 6\rangle[4|5+6|1\rangle\langle 5|1+6|2]} + \\ +\frac{\langle 4\ 5\rangle^3[6\ 1]^2[6\ 2]}{t_3^{[3]}\langle 3\ 4\rangle[1\ 2]\langle 3|4+5|6]\langle 5|1+6|2]} + \\ +\frac{\langle 1|4+5|6]\langle 2|4+5|6]^3}{t_4^{[3]}\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]\langle 3|4+5|6]\langle 1|5+6|4]} \tag{4.3}$$

Choosing p_3 and p_4 as reference momenta, we have

$$A(\Lambda_1^+\Lambda_2^- g_3^- g_4^+ g_5^+ g_6^-) = -\frac{\langle 3|4+5|1]^3\langle 3|4+5|2]}{t_3^{[3]}[6\ 1][1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 3|4+5|6][2|3+4|5\rangle} + \\ +\frac{\langle 1\ 3\rangle\langle 2\ 3\rangle^2[4\ 5]^3}{t_1^{[3]}\langle 1\ 2\rangle[5\ 6]\langle 1|2+3|4]\langle 3|1+2|6]} + \\ +\frac{[2\ 4]\langle 6|2+3|4]^3}{t_2^{[3]}[2\ 3]\langle 5\ 6\rangle[3\ 4]\langle 1|2+3|4][2|3+4|5\rangle} \tag{4.4}$$

Choosing p_2 and p_3 as reference momenta, we have

$$A(\Lambda_1^+\Lambda_2^- g_3^+ g_4^- g_5^+ g_6^-) = -\frac{[1\ 3]^3\langle 4\ 6\rangle^4}{t_4^{[3]}[1\ 2]\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6|1+2|3][1|2+3|4\rangle} + \\ +\frac{\langle 6\ 2\rangle^3[3\ 5]^4}{t_3^{[3]}\langle 1\ 2\rangle[3\ 4][4\ 5]\langle 6|5+4|3]\langle 2|3+4|5]} + \\ +\frac{\langle 4\ 2\rangle^3[1\ 5]^3\langle 4|1+6|5]}{t_2^{[3]}\langle 2\ 3\rangle\langle 3\ 4\rangle[5\ 6][6\ 1]\langle 2|3+4|5]\langle 4|3+2|1]} \tag{4.5}$$

Choosing p_2 and p_3 as reference momenta, we have

$$
\begin{aligned}
A(\Lambda_1^+\Lambda_2^- g_3^+ g_4^+ g_5^- g_6^-) = & \frac{\langle 2|3+4|1]^3}{\langle 2\ 3\rangle\langle 3\ 4\rangle[5\ 6][6\ 1]\langle 2|3+4|5][1|2+3|4\rangle} + \\
& + \frac{[3\ 4]^3\langle 6\ 2\rangle^3}{t_3^{[3]}\langle 1\ 2\rangle[4\ 5]\langle 6|5+4|3]\langle 2|3+4|5]} - \\
& - \frac{[1\ 3]^3\langle 5\ 6\rangle^3}{t_1^{[3]}[1\ 2]\langle 4\ 5\rangle\langle 6|1+2|3][1|2+3|4\rangle}
\end{aligned}
\tag{4.6}
$$

The first term in equation (4.6) seems to be a little out of place, which carries no $t^{[3]}$ factor. It turns out that this term can be merged into the other two by choosing different reference lines. Specifically, by choosing p_3 and p_4 as reference momenta, (note that lines 3 and 4 have the same helicity), we can get an equivalent but more compact result

$$
\begin{aligned}
A(\Lambda_1^+\Lambda_2^- g_3^+ g_4^+ g_5^- g_6^-) = & \frac{\langle 2|5+6|4]^3}{t_1^{[3]}\langle 1\ 2\rangle\langle 2\ 3\rangle[4\ 5][5\ 6]\langle 3|4+5|6]} - \\
& - \frac{\langle 5|4+3|1]^3}{t_3^{[3]}[6\ 1][1\ 2]\langle 3\ 4\rangle\langle 4\ 5\rangle\langle 3|4+5|6]}
\end{aligned}
\tag{4.7}
$$

These results are quite simple and compact. We would like to emphasis that forms of these results depend on the choice of reference lines, though all expressions are equivalent. There could be simpler expressions for these amplitudes if one chooses more clever reference lines.

Results in equation (4.1–4.7) agree fully with those in [29] which were obtained by conventional field theory, up to a possible overall sign,[1] but assume extremely simple and compact forms. We have also computed these six amplitudes by using extended CSW prescriptions [12], and found them to be equivalent (usually in different forms) to the ones shown above. It can also be easily checked that our results satisfy all supersymmetric Ward identities.

5. Conclusion

In this paper, we have calculated tree-level amplitudes of $g_1^- g_2^- g_3^- g_4^+ g_5^+ \cdots g_n^+$ in an application of the BCF/BFCW rules. As expected, extremely compact formulas were obtained. The BCF/BCFW recursion relations were then extended to include fermions of multi-flavors, from which MHV and $\overline{\text{MHV}}$ amplitudes are reproduced correctly. We have also calculated non-MHV amplitudes of processes with two fermions and four gluons. Results thus obtained are equivalent to those obtained by using extended CSW prescriptions, and those by conventional field theory calculations. These results provide more strong evidences

[1] In previous versions of this paper, it was claimed that our results do not agree with those in [29]. The mismatch was due to the fact that spinor products used in [29] and those we are using differ by a minus sign. Fortunately, or unfortunately, this minus sign cancels out in cases of pure gluons, which prevented it to be located for quite a while. We thank L. Dixon and C. Berger for assuring us that equation (4.5–7) are equivalent to the corresponding ones in [29], so the authors were encouraged enough to finally resolve the issue by digging into the computer codes again.

to support both the CSW and the BCF/BCFW prescriptions, as well as present extensions to include fermions. As a distinctive feature, our results assume extremely simple and compact forms and the method is elegant enough so, many difficult amplitudes can be calculated with relative easy.

Before closing, let us try to justify the extended rules by following the BCFW proof. The BCFW proof has three important ingredients: the construction of a rational function $A(z)$ of z, the assertion that $A(z)$ has only simple z-poles, and that $A(z)$ vanishes as $z \to \infty$. From which one has

$$A(z) = \sum_{i,j} \frac{c_{ij}}{z - z_{ij}} \tag{5.1}$$

where C_{ij} are independent of z. Thus, z-pole positions and residues determine the physical amplitude $A(0)$ uniquely. When fermions are included, the first two ingredients can easily be established by simply repeating reasonings of [26]. The importance of a vanishing $A(z)$ as $z \to \infty$ is obvious. Otherwise, one has instead,

$$A(z) = \sum_{i,j} \frac{c_{ij}}{z - z_{ij}} + f(z) \tag{5.2}$$

where $f(z)$ is a holomorphic function of z. When $f(0)$ has to be included to get the correct physical amplitude $A(0)$, new ways have to be devised to calculate the extra function $f(z)$.

Now we show that by choosing reference lines properly, $A(z)$ vanishes indeed as $z \to \infty$, when fermion lines are present. First, we use the MHV Feynman diagrams. In this case, $A(\infty)$ does vanish if no two adjacent fermion lines are taken as reference lines, as mentioned in section 3. Otherwise, $A(\infty)$ is finite, which can easily be verified.

If we want to make an argument independent of MHV Feynman diagrams, one has to constrain the types of reference lines properly, just as in the case of pure gluons. To see this, we first classify limiting behaviors of vertices and external and internal lines as $z \to \infty$. The behaviors of gluon vertices, gluon internal and external lines are the same as those characterized in section 3.1 of [26]. Now fermions. Fermion-fermion-gluon vertices are independent of z and fermion internal lines approach to constants at the worst. Depending on the helicity of the shifted fermion and the way it is shifted, an external fermion line would approach a constant or infinity $\sim z$. So it is possible that one may end up with a wrong $A(z)$ if reference lines are not chosen properly. For example, when two fermion lines are picked up as reference lines, $A(z)$ looks as if could approach a constant at the best and an infinity $\sim z^2$ at the worst. Experience tells that, also supported by MHV Feynman diagrams, the behavior of $A(z)$ is actually much better, though the present argument cannot prove cancellations of bad z behaviors in general. Of course, one can usually stay on the safe side, by choosing "good" reference lines. For example, one can choose two external gluons of helicities $(-,+)$ for reference, $A(z)$ would vanish $\sim 1/z^2$, as now all z factors from internal gluons and gluon vertices cancel out, due to the presence of fermion lines. If one chooses one fermion and one gluon (Λ^-, g^+) for reference, $A(z)$ would vanish $\sim 1/z$ as $z \to \infty$.

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